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An Indirect Single-Position Coordinate Determination Method Considering Motion Invariants under Singular Measurement Errors

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Abstract—The problem of indirect single-position coordinate determination based on the smoothed measurements of bearing and the radial velocity of an object is solved considering motion invariants and singular measurement errors. Such errors are represented as an appropriate linear combination with unknown spectral coefficients in a given finite-dimensional functional space. Possible application of the developed method to different models of motion and observation is considered. Analytical relations are derived for estimating accuracy characteristics and methodological errors. A comparative evaluation of computational cost is presented.

Keywords: single-position coordinate determination, indirect method, bearing, radial velocity, invariant, first integral, unbiasedness condition, invariance condition, optimality criterion, Lagrange's multiplier method, posterior variance of estimation error

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1. INTRODUCTION

Nowadays, the issues of single-position (active and passive) coordinate determination are still topical in a wide range of location and navigation problems. As a rule, these methods are implemented based on direct and indirect measurements of bearing, phase differences, Doppler frequencies, relative signal powers, and their derivatives. Also, additional information from various illumination sources, reflectors (of natural and artificial origin), external control systems, as well as a priori data on the structure and some parameters of the emitted signal, object speed, the initial or final point of its route, the presence of barrage and maneuvering areas, etc. are used. (For example, we refer to the publications [1–33].)

The solution of single-position coordinate determination problems under various types of interference fits well into the optimal Kalman estimation scheme in a stochastic formulation (as a rule, with state space expansion) with direct and pseudo measurements [9, 10, 17–26, 30, 36]. In practice, however, rather simple suboptimal indirect coordinate determination methods with smoothed measurements are often used for a wide class of problems (e.g., those related to express and post-processing of trajectory and telemetry data in range command and measurement complexes, real-time tracking of maneuvering objects, etc. [2, 7, 8, 11]). These methods are based on simple deterministic motion models (linear, piecewise linear, polynomial, piecewise polynomial, differential, piecewise differential, group, piecewise group, and many others), known analytical relationships between the estimated and measured parameters, and simple procedures for smoothing observations based on the least squares method (LSM) and its various modifications. Being inferior to optimal (linear and nonlinear) filtering methods in terms of potential accuracy, they are easy to implement in practice in real time under high-quality smoothed measurements. Furthermore, their numerical implementation causes no problems related to transients, convergence, and strict requirements for the volume and quality of initial a priori information (which is often characteristic of optimal methods, e.g., when considering the effects of "smearing accuracy" or "rigidity" [9, 32]). For instance, the complexes mentioned above traditionally involve multistage information processing; indirect methods are applied therein at the stage of express processing whereas optimal methods are usually implemented at the stage of post-processing.

Some indirect single-position coordinate determination methods do not use bearing but operate with periodic pulse radio signals and are oriented to measuring the continuous frequency bias of the received signal at the observation point due to the movement of either the radiation source or the observer [3, 7]. The fundamental disadvantage of these methods is the necessity to consider a priori information about the speed of the object (or that of the source, or that of the observer), which is often unacceptable in practice. In addition, the coordinate determination problem is limited to finding the range and heading angle within the uniform rectilinear motion model; hence, it is impossible to estimate all object location parameters for an arbitrary time instant. An attempt to eliminate the speed-related limitation was undertaken in [27]; but in this case, it is required to track the evolution of the Doppler frequency considering the continuous accumulation (counting) of pulses of the received signal at the observation point. Obviously, the matter concerns only high-speed objects and severe constraints on observation conditions, and the uniform rectilinear motion model is also used. The general drawback of the indirect methods discussed in [3, 7, 27] is the technical complexity of their practical implementation.

There are goniometric Doppler methods for the single-position determination of motion parameters (e.g., see [3, 4, 7, 9, 29]) with direct (radial velocity and bearing) and indirect (the derivatives of different orders) measurements without the a priori information mentioned above. These methods are focused on the simplest motion models (e.g., orbital) and neglect the possibility of constructing several independent coordinate determination channels and the appearance of singular primary measurement errors that devalue the information contained in indirect measurements (the derivatives of radial velocity and bearing).

Note the method [29], which operates with derivatives up to the second order inclusive and forms adaptive coordinate determination algorithms based on several parallel algorithms corresponding to the invariants of object motion. However, according to the analysis, the explicit-form relations and the corresponding algorithms obtained in [29] are dependent and redundant; they are also focused on the uniform rectilinear motion model only.

This paper develops an indirect single-position coordinate determination method invariant with respect to singular errors of a given class. (Such errors are represented as an appropriate linear combination with unknown spectral coefficients in a given finite-dimensional functional space.) Based on a complete set of invariants (for a wide class of motion models), the method forms a family of independent quasi-optimal solutions and the resulting estimate of object motion parameters using these solutions. The comparative computational gain is demonstrated.

According to [8, 11, 33, 37], invariants can effectively serve to solve a whole class of applied target problems of single- and multi-position location and navigation based on indirect methods. Here, we show the possibility of decentralization, parallelization, and reduction of computational cost in processing measurements in various-type systems based on invariants of continuous groups of Lie transformations (CGLT) and first integrals used to describe the motion of various objects.

2. PROBLEM STATEMENT

Consider an object whose motion in a separate observation area is described in the Cartesian rectangular frame by some operator equation (e.g., in the vector-algebraic or vector-differential form)

$$\mathbf{G}\,(t, \boldsymbol{\rho}, \boldsymbol{\eta}) = 0 \quad \forall\, t \in [0, T]\,, \tag{2.1}$$

where $\boldsymbol{\rho} = [x, y, z]^{\mathrm{T}}$ ($x = x(t)$, $y = y(t)$, $z = z(t)$) denotes the object's coordinate vector and $\boldsymbol{\eta}$ is the vector of unknown real-valued parameters.

Assume that the coordinates x, y, z are smooth and differential functions (a required number of times) and the vector $\boldsymbol{\rho}$ is assigned the vector of spherical coordinates $\boldsymbol{\varsigma} = [r, \lambda, \varphi]^{\mathrm{T}}$, where r, λ, and φ are inclined range, longitude, and latitude, respectively. Let $\mathbf{X} = \left[r^{(1)}, \lambda, \varphi\right]^{\mathrm{T}}$ be the vector of direct measurements, where $r^{(1)} = dr/dt$), and let $\mathbf{Y}$ be the vector of indirect measurements, whose coordinates are the derivatives of $r^{(1)}, \lambda, \varphi$ of different orders, necessary to implement a version of the method developed below. We choose a grid (sliding window, further termed the window for simplicity) $\{t_{n+i}, i = \overline{-m, m}\}$, where $n \geq m$, $m \in \{1, 2, \ldots\}$, $t_{n+i} \in [0, T]$, and $2m + 1$ is the window size. Introducing the notation $\mu \in \left\{r^{(1)}, \lambda, \varphi\right\}$, we adopt the additive observation equation

$$\mathbf{H}_\mu = \boldsymbol{\mu} + \mathbf{s}_\mu + \boldsymbol{\xi}_\mu, \tag{2.2}$$

where

$$\boldsymbol{\mu} = [\mu_{n+i}, i = \overline{-m, m}]^{\mathrm{T}}, \quad \mathbf{s}_\mu = [s_{\mu,n+i}, i = \overline{-m, m}]^{\mathrm{T}},$$
$$\boldsymbol{\xi}_\mu = [\xi_{\mu,n+i}, i = \overline{-m, m}]^{\mathrm{T}}, \quad \mu_{n+i} = \mu\,(t_{n+i}),$$
$$s_{\mu,n+i} = s_\mu\,(t_{n+i}), \quad \xi_{\mu,n+i} = \xi_\mu\,(t_{n+i}).$$

In (2.2), $s_\mu(t)$ means the singular error

$$s_\mu(t) = \mathbf{D}_\mu^{\mathrm{T}} \boldsymbol{\Theta}_\mu(t), \tag{2.3}$$

where

$\mathbf{D}_\mu = \left[d_{\mu k}, k = \overline{0, K}\right]^{\mathrm{T}}$ is the vector of unknown spectral coefficients and

$\boldsymbol{\Theta}_\mu(t) = \left[\theta_{\mu k}(t), k = \overline{0, K}\right]^{\mathrm{T}}$ is the vector of given basis functions.

The function $\mu = \mu(t)$ has the spectral representation

$$\mu(t) = \mathbf{A}_\mu^{\mathrm{T}} \boldsymbol{\Psi}_\mu(t), \tag{2.4}$$

where

$\mathbf{A}_\mu = \left[a_{\mu b}, b = \overline{0, B}\right]^{\mathrm{T}}$ is the vector of unknown coefficients and

$\boldsymbol{\Psi}_\mu(t) = \left[\psi_{\mu b}(t), b = \overline{0, B}\right]^{\mathrm{T}}$ is the vector of given basis functions.

The vector $\boldsymbol{\xi}_\mu$ consists of random errors with zero and the correlation matrix $\mathbf{K}_\mu = [k_{\mu,n+i,n+j}, i, j = \overline{-m, m}]$.

Models (2.1)–(2.4) are widely used in various localization and navigation problems. Complex trajectories (e.g., those of maneuvering objects) can be described by applying a separate model (2.1) for each observation area. In particular, a very promising approach is to describe such trajectories via the simplest groups of Lie transformations (e.g., shift, rotation, and stretching [8, 11, 31, 33–37]).

Based on the set of invariants of equation (2.1) (in particular, the first integrals of motion or the invariants of CGLT), it is required to develop an indirect coordinate determination method considering (2.2)–(2.4) and the accepted constraints with the following features: the method involves no state space expansion; the method is robust to the singular error; the method allows estimating the object's motion parameters from the extended vector of direct and indirect measurements $\mathbf{Z} = \left[\mathbf{X}^{\mathrm{T}}, \mathbf{Y}^{\mathrm{T}}\right]^{\mathrm{T}}$, whose coordinates are estimated with minimum posterior variances.

3. THE PRINCIPLE OF DETERMINING MOTION PARAMETERS BASED ON INVARIANTS

Let us associate with equation (2.1) a scalar invariant $I = I\,(t, \boldsymbol{\rho}, \boldsymbol{\gamma}_I)$, where the vector $\boldsymbol{\gamma}_I$ consists of some derivatives of the coordinates of the vector $\boldsymbol{\rho}$. On the solutions $\boldsymbol{\rho}(t)$ and $\boldsymbol{\gamma}_I(t)$ of equation (2.1), this invariant satisfies the condition

$$I\,(t, \boldsymbol{\rho}(t), \boldsymbol{\gamma}_I(t)) = C = \text{const} \quad \forall\, t \in [0, T]\,. \tag{3.1}$$

Passing to spherical coordinates in (3.1) gives

$$Q\left(t, \boldsymbol{\varsigma}(t), \boldsymbol{\gamma}_Q(t)\right) = C = \text{const} \quad \forall\, t \in [0, T]\,, \tag{3.2}$$

where the vector $\boldsymbol{\gamma}_Q$ consists of some derivatives of the coordinates of the vector $\boldsymbol{\varsigma}$.

The way to find the invariants is entirely determined by the kind of equation (2.1).

We calculate the total derivative of the left- and right-hand sides of equation (3.2):

$$\frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial \boldsymbol{\varsigma}}\left(\frac{d\boldsymbol{\varsigma}}{dt}\right)^{\mathrm{T}} + \frac{\partial Q}{\partial \boldsymbol{\gamma}_Q}\left(\frac{d\boldsymbol{\gamma}_Q}{dt}\right)^{\mathrm{T}} = 0 \quad \forall\, t \in [0, T]\,. \tag{3.3}$$

Expanding all derivatives in (3.3) yields the equation

$$W\,(t, \boldsymbol{\varsigma}, \boldsymbol{\gamma}_W) = 0 \quad \forall\, t \in [0, T]\,, \tag{3.4}$$

where the vector $\boldsymbol{\gamma}_W$ consists of all possible derivatives of r, λ, φ.

Solving this equation for r, we determine the inclined range (distance to the object):

$$r = W^{-1}\,(t, \mathbf{Z})\,. \tag{3.5}$$

Associating with equation (2.1) the set of independent invariants $I_l = I_l\,(t, \boldsymbol{\rho}, \boldsymbol{\gamma}_I)$, $l = \overline{1, L}$, by analogy with (3.1)–(3.5), we obtain the set of formulas

$$r\,[l] = W_{[l]}^{-1}\left(t, \mathbf{Z}_{[l]}\right), \quad l = \overline{1, L}. \tag{3.6}$$

This set can be used in an adaptive version of the inclined range estimation procedure in order to improve the accuracy of estimation considering measurement errors. For example, if for a fixed time instant t, the vector $\mathbf{Z}_{[l]}$ is estimated with an error characterized by zero mean the correlation matrix $\mathbf{K}_{\mathbf{Z}}$, then the variance of the inclined range estimate is given by

$$\sigma^2_{r[l]} = \mathbf{H}_{[l]}^{\mathrm{T}} \mathbf{K}_{\mathbf{Z}[l]} \mathbf{H}_{[l]}, \quad l = \overline{1, L}, \tag{3.7}$$

where the column vector $\mathbf{H}_{[l]}$ consists of the partial derivatives of (3.6) with respect to the elements of the vector $\mathbf{Z}_{[l]}$ calculated on their mathematical expectations.

As an optimal version of the inclined range estimation procedure we select the one for which

$$l^* = \arg\min_{l} \sigma^2_{r[l]}, \quad l^* \in \{1, 2, \ldots, L\}\,. \tag{3.8}$$

The Cartesian coordinates of the object can be determined using the dependencies

$$x\,[l^*] = r\,[l^*] \cos\varphi \cos\lambda, \quad y\,[l^*] = r\,[l^*] \cos\varphi \sin\lambda, \quad z\,[l^*] = r\,[l^*] \sin\varphi, \tag{3.9}$$

where the angular coordinates λ and φ are replaced by either direct measurements or their smoothed values.

Generally, we can use the set of probable models $\mathbf{G}_k(t, \boldsymbol{\rho}, \boldsymbol{\eta}) = 0$, $k = \overline{0, K}$, for a given observation area instead of (2.1). In this case, the algorithm (3.7)–(3.9) takes the form

$$\begin{cases} \sigma^2_{r[k,l]} = \mathbf{H}^{\mathrm{T}}_{[k,l]} \mathbf{K}_{\mathbf{Z}[k,l]} \mathbf{H}_{[k,l]}, \quad k = \overline{1, K}, \quad l = \overline{1, L_k}, \\ [k^*, l^*] = \arg\min\limits_{[k,l]} \sigma^2_{r[k,l]}, \quad k^* \in \{1, 2, \ldots, K\}, \quad l^* \in \{1, 2, \ldots, L_k\}, \\ x[k^*, l^*] = r[k^*, l^*] \cos\varphi \cos\lambda, \\ y[k^*, l^*] = r[k^*, l^*] \cos\varphi \sin\lambda, \\ z[k^*, l^*] = r[k^*, l^*] \sin\varphi. \end{cases} \tag{3.10}$$

The algorithm (3.10) parallelizes the computational process considering the number of invariants used and adapts the estimation procedure of the object motion parameters to the observation conditions.

4. DESIGN AND USE OF INVARIANTS: SOME EXAMPLES

Consider a separate observation area in which the following general CGLT [32–34] corresponds to equation (2.1):

$$T_a: \quad \boldsymbol{\rho}' = \mathbf{f}(a, \boldsymbol{\rho}_0, \boldsymbol{\eta}_0) \quad \forall\, a \in \Delta_a \subset R^1, \tag{4.1}$$

where $\boldsymbol{\rho}' = [x', y', z']^{\mathrm{T}}$, $\mathbf{f}(a, \boldsymbol{\rho}_0, \boldsymbol{\eta}_0) = [f_x, f_y, f_z]^{\mathrm{T}}$, $\boldsymbol{\eta}_0$ is the vector of numerical parameters of the group and a is a real-valued group parameter such that $\mathbf{f}(a_0, \boldsymbol{\rho}, \boldsymbol{\eta}_0) = \boldsymbol{\rho}$ for $a = a_0$, $a_0 \in \Delta_a$.

Model (4.1) describes the object's trajectory; when treating the group parameter as a time-varying function $a = a(t, \boldsymbol{\chi}_0)$, where $\boldsymbol{\chi}_0$ is the vector of generally unknown numerical parameters, we can describe the time law of motion along this trajectory. With the change of coordinates $\boldsymbol{\rho}' = \boldsymbol{\rho}$, $\boldsymbol{\rho} = \boldsymbol{\rho}_0$, due to (2.1), it follows that $\boldsymbol{\rho} - \mathbf{f}(a, \boldsymbol{\rho}_0, \boldsymbol{\eta}_0) = \mathbf{G}(t, \boldsymbol{\rho}, \boldsymbol{\eta})$, $\boldsymbol{\eta} = [\boldsymbol{\rho}_0, \boldsymbol{\eta}_0, \boldsymbol{\chi}_0]^{\mathrm{T}}$.

The invariants $I = I(\boldsymbol{\rho}, \boldsymbol{\eta}_0)$ of model (4.1), independent of the parameters t, $\boldsymbol{\rho}_0$, and $\boldsymbol{\chi}_0$, are found by solving the linear partial differential equation

$$XI(\boldsymbol{\rho}, \boldsymbol{\eta}_0) = \phi_x \frac{\partial I}{\partial x} + \phi_y \frac{\partial I}{\partial y} + \phi_z \frac{\partial I}{\partial z} = 0, \tag{4.2}$$

where $X = \phi_x \partial/\partial x + \phi_y \partial/\partial y + \phi_z \partial/\partial z$ denotes the infinitesimal CGLT operator. Its coordinates are given by $\phi_x = \partial f_x/\partial a$, $\phi_y = \partial f_y/\partial a$, and $\phi_z = \partial f_z/\partial a$ at the point $a = a_0$.

In view of (4.2), an extended operator and the corresponding partial differential equation are constructed to find invariants considering the temporal nature of motion along the trajectory (4.1) and various derivatives of the vector $\boldsymbol{\rho}$; for details, see [8, 11, 31–34].

This method will be demonstrated on an example of a shift group. Let

$$T_{a=t}: \boldsymbol{\rho}' = \boldsymbol{\rho} + \boldsymbol{\eta}_0 t \;\forall\, a = t \in \Delta_a = [0, T] \subset R^1,$$

where $\boldsymbol{\eta}_0 = \mathbf{V}_0 = [V_{x0}, V_{y0}, V_{z0}]^{\mathrm{T}}$ is the velocity vector of an object moving straight and uniformly. In this case, we have $\phi_x = V_{x0}$, $\phi_y = V_{y0}$, $\phi_z = V_{z0}$, and two independent invariants, $I_{[1]} = xV_{y0} - yV_{x0} = xy^{(1)} - yx^{(1)}$ and $I_{[2]} = xV_{z0}zV_{x0} = xz^{(1)} - zx^{(1)}$. In addition, due to (3.1), $\gamma_{I[1]}(t) = \gamma_{I[1]} = \left[x^{(1)}, y^{(1)}\right]^{\mathrm{T}}$ and $\gamma_{I[2]}(t) = \gamma_{I[2]} = \left[x^{(1)}, z^{(1)}\right]^{\mathrm{T}}$. Using (3.2) and straightforward but cumbersome transformations, we obtain

$$Q_{[1]}\left(t, \varsigma(t), \boldsymbol{\gamma}_{Q[1]}(t)\right) = r^2 \lambda^{(1)} \cos^2\varphi,$$

$$Q_{[2]}\left(t, \varsigma(t), \boldsymbol{\gamma}_{Q[2]}(t)\right) = r^2 \left(\varphi^{(1)} \cos\lambda + \lambda^{(1)} \sin\lambda \sin\varphi \cos\varphi\right),$$

where

$$\boldsymbol{\gamma}_{Q[1]} = \left[\varphi^{(1)}\right] \text{ and } \boldsymbol{\gamma}_{Q[2]} = \left[\lambda^{(1)}, \varphi^{(1)}\right]^{\mathrm{T}}.$$

In view of (3.3) and (3.4), it follows that

$$W_{[1]} = 2r^{(1)}\lambda^{(1)} + r\left(\lambda^{(2)} - 2\lambda^{(1)}\varphi^{(1)}\tan\varphi\right),$$
$$W_{[2]} = 2r^{(1)}\varphi^{(1)} + r\left(\varphi^{(2)} + \left(\lambda^{(1)}\right)^2 \sin\varphi\cos\varphi\right),$$

where

$$\boldsymbol{\gamma}_{W[1]} = \left[r^{(1)}, \lambda^{(1)}, \lambda^{(2)}, \varphi^{(1)}\right]^{\mathrm{T}} \text{ and } \boldsymbol{\gamma}_{W[1]} = \left[r^{(1)}, \lambda^{(1)}, \varphi^{(1)}, \varphi^{(2)}\right]^{\mathrm{T}}.$$

Finally, concretizing (3.5) and (3.6) yields two independent formulas for the inclined range:

$$r\,[1] = \frac{2r^{(1)}\lambda^{(1)}\cos\varphi}{2\lambda^{(1)}\varphi^{(1)}\sin\varphi - \lambda^{(2)}\cos\varphi}, \tag{4.3}$$

$$r\,[2] = -\frac{2r^{(1)}\varphi^{(1)}}{\varphi^{(2)} + \left(\lambda^{(1)}\right)^2 \sin\varphi\cos\varphi}. \tag{4.4}$$

In the special cases $\varphi = \varphi^{(1)} = \varphi^{(2)} = 0$ and $\lambda = \lambda^{(1)} = \lambda^{(2)} = 0$, the expressions (4.3) and (4.4) directly imply the well-known ranging formulas

$$r\,[1] = -2r^{(1)}\lambda^{(1)}/\lambda^{(2)}, \quad r\,[2] = -2r^{(1)}\varphi^{(1)}/\varphi^{(2)}.$$

(For example, see the differential-geometrical method [4].)

Other ranging formulas can be derived using three new invariants, $I_{[3]} = x^{(1)} = V_{x0}$, $I_{[4]} = y^{(1)} = V_{y0}$, and $I_{[5]} = z^{(1)} = V_{z0}$. They lead to the independent ranging formulas

$$r\,[3] = \frac{r^{(2)}\cos\varphi - 2r^{(1)}\varphi^{(1)}\sin\varphi}{\varphi^{(2)}\sin\varphi + \left[\left(\lambda^{(1)}\right)^2 + \left(\varphi^{(1)}\right)^2\right]\cos\varphi}, \tag{4.5}$$

$$r\,[4] = \frac{r^{(2)}\sin\varphi + 2r^{(1)}\varphi^{(1)}\cos\varphi}{\left(\varphi^{(1)}\right)^2\sin\varphi - \varphi^{(2)}\cos\varphi}, \tag{4.6}$$

$$r\,[5] = \frac{r^{(2)}}{\left(\varphi^{(1)}\right)^2 + \left(\lambda^{(1)}\right)^2\cos^2\varphi}. \tag{4.7}$$

In contrast to [29], the set of formulas (4.3)–(4.7) is necessary and sufficient for constructing a parallel independent adaptive ranging algorithm, and the resulting relations are written in a compact (nonredundant) form.

Remark 1. There is no complete coincidence of the sets of measured parameters in all formulas (4.3)–(4.7). In view of (3.7)–(3.10), it is therefore possible to organize five independent channels for range calculation and adaptation to variable observation conditions.

Remark 2. The longitude λ is not explicitly included in any of the formulas; hence, the constant systematic errors in the measurements of the coordinate λ can be effectively dealt with. The latitude φ explicitly figures in all the formulas.

Remark 3. For more complex motion models with general CGLT, all possible invariants corresponding to the trajectory and the object's motion law along this trajectory, as well as independent expressions for determining the inclined range, can be found similar to the shift group.

Remark 4. For a maneuvering object, it is necessary to use a compound model based on an admissible set of a particular CGLT (e.g., shift, rotation, and stretching). An appropriate particular CGLT in some observation area is chosen by solving the identification problem with minimizing a decision function (e.g., the residual of the least squares method). Such an approach using the rotation group was considered in [31]; the object trajectory was approximated by pieces of circles of different radii.

If model (2.1) is some differential equation, then all possible invariants in the dynamic case can be found within the well-known theory of group analysis [34–36]. In practice, however, it often suffices to use particular invariants of motion, i.e., the so-called first integrals of the differential equation. We will demonstrate this approach on an example of circular orbital motion: $\mathbf{G}\left(t, \boldsymbol{\rho}, \boldsymbol{\eta}\right) = \boldsymbol{\rho}^{(2)} + \eta_0 R_0^{-3} \boldsymbol{\rho} = 0$, where $\boldsymbol{\eta} = [R_0, \eta_0]^{\mathrm{T}}$ and R_0 and η_0 are the radius and gravitational parameter of the Earth, respectively. As is well known, the invariants (first integrals) of this motion are $I_{[1]} = xy^{(1)} - yx^{(1)}$ and $I_{[2]} = xz^{(1)} - zx^{(1)}$, identical in form to the shift group invariants discussed above. However, the derivatives $x^{(1)}$, $y^{(1)}$, and $z^{(1)}$ here are not constants and the invariants $I_{[3]} = x^{(1)} = V_{x0}$, $I_{[4]} = y^{(1)} = V_{y0}$, and $I_{[5]} = z^{(1)} = V_{z0}$ used previously become inapplicable. With this fact in mind, we accept only the expressions (4.3) and (4.4) as ranging formulas in the dynamic case.

Remark 5. The single-position indirect method developed in this paper can be generalized to the class of stochastic models, for which the application of classical invariants is often very limited. At the same time, it is possible to use the so-called ε-invariants [37]. Within this approach, the invariance condition holds approximately (with accuracy up to ε), and the coordinate determination problem can be solved approximately as well.

5. CONSIDERATION OF FLUCTUATING MEASUREMENT ERRORS

We take an example of the shift group and the condition $\varphi = \varphi^{(1)} = \varphi^{(2)} = 0$ to demonstrate the implementation of the algorithm (3.7), (3.8). Clearly, in this particular case, the entire set of formulas (4.3)–(4.7) reduces to the two informative ones:

$$r\,[1] = -2r^{(1)}\lambda^{(1)}/\lambda^{(2)}, \quad r\,[2] = -r^{(2)}/\left(\lambda^{(1)}\right)^2.$$

Accordingly, we have two vectors of measured parameters: $\mathbf{Z}_{[1]} = \left[r^{(1)}, \lambda^{(1)}, \lambda^{(2)}\right]^{\mathrm{T}}$ and $\mathbf{Z}_{[2]} = \left[r^{(2)}, \lambda^{(1)}\right]^{\mathrm{T}}$. Let the matrices $\mathbf{K}_{\mathbf{Z}[1]}$ and $\mathbf{K}_{\mathbf{Z}[2]}$ be diagonal, i.e.,

$$\mathbf{K}_{\mathbf{Z}[1]} = \mathrm{diag}\left[\sigma^2_{r^{(1)}}, \sigma^2_{\lambda^{(1)}}, \sigma^2_{\lambda^{(2)}}\right] \text{ and } \mathbf{K}_{\mathbf{Z}[2]} = \mathrm{diag}\left[\sigma^2_{r(2)}, \sigma^2_{\lambda(1)}\right].$$

(With this supposition, the presentation below will be less cumbersome.) Due to $x = r\cos\lambda$, $y = r\sin\lambda$, and (3.7), we find

$$\sigma^2_{r[1]} = 4\left(\lambda^{(2)}\right)^{-2}\left\{\left(\lambda^{(1)}\right)^2\sigma^2_{r^{(1)}} + \left(r^{(1)}\right)^2\left[\sigma^2_{\lambda^{(1)}} + \left(\lambda^{(1)}/\lambda^{(2)}\right)^2\sigma^2_{\lambda^{(2)}}\right]\right\}, \tag{5.1}$$

$$\sigma^2_{r[2]} = \left(\lambda^{(1)}\right)^{-4}\left[\sigma^2_{r^{(2)}} + 4\left(r^{(2)}\right)^2\left(\lambda^{(1)}\right)^{-4}\sigma^2_{\lambda^{(1)}}\right]. \tag{5.2}$$

The priority is given to the ranging formula for which

$$l^* = \arg\min_l \sigma^2_{r[l]}, \quad l^* \in \{1, 2\}. \tag{5.3}$$

According to (5.1)–(5.3), the method involves derivatives up to the second order inclusive and can be effectively applied only on smoothed measurements. In addition, the class of high-speed objects is considered: the necessary increment of angular coordinates and radial velocity on a given observation interval must be provided [29].

6. AN AUTO-COMPENSATION ALGORITHM FOR SMOOTHING PRIMARY MEASUREMENTS

In view of (2.1)–(2.4), we consider an auto-compensation unbiased smoothing algorithm for the parameter $\mu \in \left\{r^{(1)}, \lambda, \varphi\right\}$ and its derivatives $\mu^{(q)}$, $q \in \{0, 1, 2\}$, at a point t_n using the window $\{t_{n+i}, i = \overline{-m, m}\}$. Let us rest on the general approach to estimating the values of linear functionals; for details, see [38, 39].

Within this approach, the estimate $\mu^{(q)*}$ of $\mu^{(q)}$ has the form

$$\mu^{(q)*} = \mathbf{P}_{\mu q}^{\mathrm{T}} \mathbf{H}_{\mu}, \tag{6.1}$$

where $\mathbf{P}_{\mu q} = [p_{\mu q, n+i}, i = \overline{-m, m}]^{\mathrm{T}}$ is the vector of unknown weight coefficients assigned by minimizing the variance $\sigma_{\mu q}^2$ of the estimate $\mu^{(q)*}$.

This estimate belongs to the linear class; therefore,

$$\sigma_{\mu q}^2 = \mathbf{P}_{\mu q}^{\mathrm{T}} \mathbf{K}_{\mu} \mathbf{P}_{\mu q}. \tag{6.2}$$

Furthermore, we require the unbiasedness conditions of the estimate ($\mu^{(q)} - \mathbf{P}_{\mu q}^{\mathrm{T}} \boldsymbol{\mu} = 0$) and its invariance with respect to the singular error ($\mathbf{P}_{\mu q}^{\mathrm{T}} \mathbf{s}_{\mu} = 0$). The constrained optimization problem is solved using Lagrange's multiplier method with the decision function

$$J\left(\mathbf{P}_{\mu q}, \boldsymbol{\zeta}_{\mu q}, \boldsymbol{\omega}_{\mu q}\right) = \mathbf{P}_{\mu q}^{\mathrm{T}} \mathbf{K}_{\mu} \mathbf{P}_{\mu q} + \boldsymbol{\zeta}_{\mu q}^{\mathrm{T}} \Theta_{\mu}^{\mathrm{T}} \mathbf{P}_{\mu q} + \left[\left(\boldsymbol{\Psi}_{\mu}^{\mathrm{T}}\right)^{(q)} - \mathbf{P}_{\mu q}^{\mathrm{T}} \boldsymbol{\Psi}_{\mu}\right] \boldsymbol{\omega}_{\mu q}, \tag{6.3}$$

where $\boldsymbol{\zeta}_{\mu q}$ and $\boldsymbol{\omega}_{\mu q}$ are the column vectors of the Lagrange multipliers, $\boldsymbol{\Theta}_{\mu} = [\theta_{\mu k}(t_{n+i}), i = \overline{-m, m}$, $k = \overline{0, K}\,]$ is the basis matrix of the singular error, and $\boldsymbol{\Psi}_{\mu} = \left[\psi_{\mu b}\left(t_{n+i}\right),\ i = \overline{-m, m},\ b = \overline{0, B}\right]$ is the basis matrix of the parameter $\mu = \mu(t)$.

The vector $\mathbf{P}_{\mu q}$ minimizing $\sigma_{\mu q}^2$ subject to the unbiasedness and invariance conditions has the form

$$\mathbf{P}_{\mu q} = \boldsymbol{\Lambda}_{\mu} \mathbf{K}_{\mu}^{-1} \boldsymbol{\Psi}_{\mu} \left(\boldsymbol{\Psi}_{\mu}^{\mathrm{T}} \boldsymbol{\Lambda}_{\mu} \mathbf{K}_{\mu}^{-1} \boldsymbol{\Psi}_{\mu}\right)^{-1} \boldsymbol{\Psi}_{\mu n}^{(q)}, \tag{6.4}$$

where $\boldsymbol{\Lambda}_{\mu} = E_{2m+1} - \mathbf{K}_{\mu}^{-1} \Theta_{\mu} \left(\Theta_{\mu}^{\mathrm{T}} \mathbf{K}_{\mu}^{-1} \Theta_{\mu}\right)^{-1} \Theta_{\mu}^{\mathrm{T}}$, E_{2m+1} is an identity matrix of dimensions $(2m+1) \times (2m+1)$, and $\boldsymbol{\Psi}_{\mu n}^{(q)} = d^q \boldsymbol{\Psi}_{\mu}(t)/dt^q|_{t=t_n}$.

The variance of the estimate $\mu^{(q)*}$ is given by

$$\sigma_{\mu q}^2 = \left(\boldsymbol{\Psi}_{\mu n}^{(q)}\right)^{\mathrm{T}} \left[\left(\mathbf{K}_{\mu}^{-1} \boldsymbol{\Psi}_{\mu}\right)^{\mathrm{T}} \left(\boldsymbol{\Lambda}_{\mu}\right)^{\mathrm{T}} \boldsymbol{\Psi}_{\mu}\right]^{-1} \mathbf{H}_{\mu} \left(\boldsymbol{\Psi}_{\mu}^{\mathrm{T}} \boldsymbol{\Lambda}_{\mu} \mathbf{K}_{\mu}^{-1} \boldsymbol{\Psi}_{\mu}\right)^{-1} \boldsymbol{\Psi}_{\mu n}^{(q)}, \tag{6.5}$$

where

$$\mathbf{H}_{\mu} = \left(\mathbf{K}_{\mu}^{-1} \boldsymbol{\Psi}_{\mu}\right)^{\mathrm{T}} \left(\boldsymbol{\Lambda}_{\mu}\right)^{\mathrm{T}} \mathbf{K}_{\mu} \boldsymbol{\Lambda}_{\mu} \mathbf{K}_{\mu}^{-1} \boldsymbol{\Psi}_{\mu}.$$

Clearly, the methodological error due to neglecting the tail of the series (2.4) has the mathematical expectation

$$\varepsilon_{\mu q} = \Delta_{\mu n}^{(q)} - \mathbf{P}_{\mu q}^{\mathrm{T}} \boldsymbol{\Delta}_{\mu n}, \tag{6.6}$$

where $\Delta_\mu = \Delta_\mu(t)$ is the series tail and $\Delta_{\mu n}^{(q)}$ denotes its qth derivative at the point $t = t_n$, $\mathbf{\Delta}_{\mu n} = \left[\Delta_\mu\left(t_{n+i}\right),\ i = \overline{-m, m}\right]^{\mathrm{T}}$.

According to [38, p. 62], with increasing the number of spectral coefficients in the singular error model (2.3), the algorithm (6.1)–(6.6) reduces computational cost by 47% compared to the traditional extended least-squares method. As a result, the smoothing problem is solved faster.

Considering (6.1)–(6.6), we can construct the desired estimates of the object motion parameters invariant to singular measurement errors. For example, formulas (4.3) and (4.4) yield the following robust estimates of the inclined range for two invariants:

$$r[1] = \frac{2\left(\mathbf{P}_{r1}^{\mathrm{T}}\mathbf{H}_r\right)\left(\mathbf{P}_{\lambda 1}^{\mathrm{T}}\mathbf{H}_\lambda\right)\cos\left(\mathbf{P}_{\varphi 0}^{\mathrm{T}}\mathbf{H}_\varphi\right)}{2\left(\mathbf{P}_{\lambda 1}^{\mathrm{T}}\mathbf{H}_\lambda\right)\left(\mathbf{P}_{\varphi 1}^{\mathrm{T}}\mathbf{H}_\varphi\right)\sin\left(\mathbf{P}_{\varphi 0}^{\mathrm{T}}\mathbf{H}_\varphi\right) - \left(\mathbf{P}_{\lambda 2}^{\mathrm{T}}\mathbf{H}_\lambda\right)\cos\left(P_{\varphi 0}^{\mathrm{T}}\mathbf{H}_\varphi\right)}, \tag{6.7}$$

$$r[2] = -\frac{2\left(\mathbf{P}_{r1}^{\mathrm{T}}\mathbf{H}_r\right)\left(\mathbf{P}_{\varphi 1}^{\mathrm{T}}\mathbf{H}_\varphi\right)}{\left(P_{\varphi 2}^{\mathrm{T}}\mathbf{H}_\varphi\right) + \left(\mathbf{P}_{\lambda 1}^{\mathrm{T}}\mathbf{H}_\lambda\right)^2\sin\left(\mathbf{P}_{\varphi 0}^{\mathrm{T}}\mathbf{H}_\varphi\right)\cos\left(\mathbf{P}_{\varphi 0}^{\mathrm{T}}\mathbf{H}_\varphi\right)}. \tag{6.8}$$

The ranges for the variants (4.5)–(4.7) and the Cartesian coordinates (3.9) of the observed object are determined by analogy with (6.7) and (6.8).

According to the results of computational experiments [38, 39], the auto-compensation smoothing algorithm demonstrates high effectiveness in anomalous measurement conditions. Hence, it is possible to form stable estimates of the derivatives of the radial velocity and angular coordinates necessary for the successful application of the single-position indirect coordinate determination method. Simulation results for the adaptive algorithm (3.7), (3.8) in the case of rectilinear uniform object motion were presented in [29]. They show that the method is applicable to high-precision measurements, while the reliability of coordinate determination significantly depends on the object's dynamics and observation conditions.

7. CONCLUSIONS

The method developed above considerably expands the scope of quasi-optimal indirect fast estimation methods robust to singular measurement errors and observation conditions of high-speed objects for their single-position coordinate determination. This method can be effectively used as a tool for intelligent and analytical improvement of the existing and next-generation single-position systems of active and passive location and navigation, independently or in combination with traditional statistical methods (e.g., least squares, maximum likelihood, maximum posterior probability density, and dynamic filtering).

The method has limitations on the classes of single-position systems in terms of measurement accuracy, observation conditions, and the types of objects to be tracked.

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